Trigonometric Identities

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$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\dots$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\tanh\theta = \frac{1 - e^{-2\theta}}{1 + e^{-2\theta}} \qquad \coth\theta = \frac{1 + e^{-2\theta}}{1 - e^{-2\theta}}$$

Pythagorean

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Sum-Difference

$$\sin(\theta \pm \psi) = \sin\theta\cos\psi \pm \cos\theta\sin\psi$$
$$\cos(\theta \pm \psi) = \cos\theta\cos\psi \mp \sin\theta\sin\psi$$
$$\tan(\theta \pm \psi) = \frac{\tan\theta \pm \tan\psi}{1 \mp \tan\theta\tan\psi}$$

Periodicity

$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$	$\sin(\theta + \pi) = -\sin(\theta)$	$\sin(\theta + 2\pi) = \sin(\theta)$
$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta)$	$\cos(\theta + \pi) = -\cos(\theta)$	$\cos(\theta + 2\pi) = \cos(\theta)$
$\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$	$\tan(\theta + \pi) = \tan(\theta)$	$\tan(\theta + 2\pi) = \tan(\theta)$

Sum to Product

$$\sin\theta \pm \sin\psi = 2\sin\left(\frac{\theta \pm \psi}{2}\right)\cos\left(\frac{\theta \mp \psi}{2}\right)$$
$$\cos\theta + \cos\psi = 2\cos\left(\frac{\theta + \psi}{2}\right)\cos\left(\frac{\theta - \psi}{2}\right)$$
$$\cos\theta - \cos\psi = -2\sin\left(\frac{\theta + \psi}{2}\right)\sin\left(\frac{\theta - \psi}{2}\right)$$

Power Reduction

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$
$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Product to Sum

$$\sin\theta\sin\psi = \tfrac{1}{2}(\cos(\theta - \psi) - \cos(\theta + \psi))$$
$$\cos\theta\cos\psi = \tfrac{1}{2}(\cos(\theta - \psi) + \cos(\theta + \psi))$$
$$\sin\theta\cos\psi = \tfrac{1}{2}(\sin(\theta - \psi) + \sin(\theta + \psi))$$
$$\cos\theta\sin\psi = \tfrac{1}{2}(\sin(\theta + \psi) - \sin(\theta - \psi))$$

Half Angle

$$\sin\left(\tfrac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$
$$\cos\left(\tfrac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$
$$\tan\left(\tfrac{\theta}{2}\right) = \csc\theta - \cot\theta = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

Double Angle

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$$

Composition

$$\sin(\arccos(x)) = \sqrt{1 - x^2} \qquad \sin(\arctan(x)) = \frac{x}{\sqrt{1 + x^2}}$$
$$\cos(\arcsin(x)) = \sqrt{1 - x^2} \qquad \cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}}$$
$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1 - x^2}} \qquad \tan(\arccos(x)) = \frac{\sqrt{1 - x^2}}{x}$$

Pythagorean

$$\cosh^2\theta - \sinh^2\theta = 1$$
$$1 - \tanh^2\theta = \operatorname{sech}^2\theta$$
$$\coth^2\theta - 1 = \operatorname{csch}^2\theta$$

Sum-Difference

$$\sinh(\theta \pm \psi) = \sinh\theta\cosh\psi \pm \cosh\theta\sinh\psi$$
$$\cosh(\theta \pm \psi) = \cosh\theta\cosh\psi \pm \sinh\theta\sinh\psi$$
$$\tanh(\theta \pm \psi) = \frac{\tanh\theta \pm \tanh\psi}{1 \pm \tanh\theta\tanh\psi}$$

Double Angle

$$\sinh(2\theta) = 2\sinh\theta\cosh\theta$$
$$\tanh(2\theta) = \frac{2\tanh\theta}{1 + \tanh^2\theta}$$
$$\cosh(2\theta) = \cosh^2\theta + \sinh^2\theta = 2\cosh^2\theta - 1 = 1 + 2\sinh^2\theta$$

Inverse

$$\operatorname{asinh} r = \ln\left(r + \sqrt{r^2 + 1}\right) \qquad \operatorname{atanh}(-1 < r < 1) = \tfrac{1}{2}\ln\left(\frac{1 + r}{1 - r}\right)$$
$$\operatorname{acosh}(r \geq 1) = \ln\left(r + \sqrt{r^2 - 1}\right) \qquad \operatorname{acoth}(|r| > 1) = \tfrac{1}{2}\ln\left(\frac{1 + r}{1 - r}\right)$$
$$\operatorname{asech}(0 < r < 1) = \ln\left(\frac{1 + \sqrt{1 - r^2}}{r}\right) \qquad \operatorname{acsch} r = \ln\left(\frac{1}{r} + \frac{\sqrt{1 + r^2}}{|r|}\right)$$

Derivatives

$\frac{d}{dx}$ Trig	=	$\frac{d}{dx}$ Hyperbolic	=
$\tan x$	$\sec^2 x$	$\tanh x$	$\operatorname{sech}^2 x$
$\cot x$	$-\csc^2 x$	$\coth x$	$-\operatorname{csch}^2 x$
$\sec x$	$\sec(x)\tan(x)$	$\operatorname{sech} x$	$-\operatorname{sech}(x)\tanh(x)$
$\csc x$	$-\csc(x)\cot(x)$	$\operatorname{csch} x$	$-\operatorname{csch}(x)\coth(x)$
$\operatorname{asin} x$	$\frac{1}{\sqrt{1 - x^2}}$	$\operatorname{asinh}\frac{x}{a}$	$\frac{1}{\sqrt{x^2 + a^2}}$
$\operatorname{acos} x$	$-\frac{1}{\sqrt{1 - x^2}}$	$\operatorname{acosh}\frac{x}{a}$	$\frac{1}{\sqrt{x^2 - a^2}}$
$\operatorname{atan} x$	$\frac{1}{1 + x^2}$	$\operatorname{atanh}\frac{x}{a}$ for $x^2 < a^2$	$\frac{a}{a^2 - x^2}$
$\operatorname{acot} x$	$-\frac{1}{1 + x^2}$	$\operatorname{acoth}\frac{x}{a}$ for $x^2 > a^2$	$\frac{-a}{x^2 - a^2}$
$\operatorname{asec} x$	$\frac{1}{\lvert x\rvert\sqrt{1 - x^2}}$	$\operatorname{asech} x$	$\frac{\pm 1}{\lvert x\rvert\sqrt{1 - x^2}}$
$\operatorname{acsc} x$	$-\frac{1}{\lvert x\rvert\sqrt{1 - x^2}}$	$\operatorname{acsch} x$	$-\frac{1}{\lvert x\rvert\sqrt{1 + x^2}}$

Periodicity

$$\sinh(\theta + 2\pi j) = \sinh(\theta)$$
$$\cosh(\theta + 2\pi j) = \cosh(\theta)$$
$$\tanh(\theta + \pi j) = \tanh(\theta)$$

Integrals

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$$
$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$
$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a}$$
$$\int x\sin^2(ax)\,dx = \frac{x^2}{4} - \frac{x}{4a}\sin(2ax) - \frac{1}{8a^2}\cos(2ax)$$
$$\int \sin(a_1x)\sin(a_2x)\,dx = \frac{\sin[(a_1 - a_2)x]}{2(a_1 - a_2)} - \frac{\sin[(a_1 + a_2)x]}{2(a_1 + a_2)} \qquad (|a_1| \neq |a_2|)$$
$$\int \frac{1}{\cos(ax) \pm \sin(ax)}\,dx = \frac{1}{a\sqrt{2}}\log\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right|$$